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Stabilization of Oscillations in an Autonomous Corrected Conservative System by Constructing an Attracting Cycle

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Abstract—This paper considers a conservative system admitting a family of single-frequency oscillations with a domain Ω. For the original system, an autonomous controlled (ε-corrected) system with a small gain is introduced; a given oscillation from the domain Ω is stabilized by constructing a cycle that attracts all trajectories from this domain together with its ε-neighborhood. A universal adaptive control law, acting as a nonlinear force linear in velocity, is designed to track the current value of potential energy during motion. The cycle is constructed for any system oscillation. As a result, a new class of autonomous controlled systems is obtained based on the conservative system, and the operating modes of this class are stabilized (in the large) cycles with any desired energy. Examples are provided.

Keywords: conservative system, single-frequency oscillation, universal adaptive control, feedback, potential energy tracking, attracting cycle, stabilization

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1. INTRODUCTION

This paper is devoted to the single-frequency oscillations (periodic motions) of a conservative system. They form families by a parameter, i.e., the constant energy h. The families are divided into nondegenerate and degenerate. On a nondegenerate family, the period $T(h)$ varies monotonically with the constant h; an example is a family of pendulum oscillations. On a degenerate family, oscillations are isochronous.

Nondegenerate oscillations can always be continued [1, 2] to a global family of oscillations. In a conservative system, a global family is described by a reduced conservative system with one degree of freedom. The same result holds for a family of degenerate oscillations of a conservative system, but this issue is not considered separately in the paper. In any case, the problem of investigating a conservative system with one degree of freedom arises first.

The idea of oscillation control by constructing a limit cycle goes back to L.S. Pontryagin [3], who found necessary and sufficient conditions for isolating a limit cycle from a family of periodic solutions of a Hamiltonian system by using non-Hamiltonian perturbations. B. van der Pol [4] investigated the relaxation modes of a regerative receiver (in the absence of a perturbing force). In 1929, A.A. Andronov discovered that the stable self-oscillations constructed by him and independently by van der Pol are a physical embodiment of Poincaré curves. In 1881, A. Poincaré [5, Chap. VI] introduced the concept of a limit cycle for a system in the plane. The theory of self-oscillations was developed in Andronov's school [6].

Poincaré singled out the isolated case when periodic solutions are preserved qualitatively with varying the parameter. In the non-isolated cases, a bifurcation occurs; here, note the main results by N.N. Bogolyubov [7], I.G. Malkin [8], and V.K. Melnikov [9]. They were continued in many directions, including, e.g., nonsmooth dynamics (see the review [10]).

In an autonomous ε-perturbed system, a bifurcation gives birth to a cycle. In this case, a Jordan cell of zero characteristic exponents (CEs) splits: one zero CE is preserved whereas the other CE becomes $-\varepsilon\alpha$. In the case $\alpha > 0$, the Andronov–Witt theorem [11] on the stability of a periodic motion of an autonomous system is valid. The concept of a cycle was also applied to multidimensional systems [12]; the Andronov–Witt theorem remains valid for them as well. The formula for computing the number α was given in [13].

In a linear periodic system, a CE is a root of the characteristic equation of the Lyapunov reduced system with constant coefficients.

In an autonomous system, the solution is determined within a shift of the initial point along the trajectory. Therefore, in the Andronov–Witt theorem, when passing to the neighborhood of a periodic motion for an n-order system, the problem of asymptotic stability is posed in the Rumyantsev sense [14] with respect to $n-1$ variables (the deviations from this motion). The solution is obtained for CEs with negative real parts; the single number $-\varepsilon\alpha$ is calculated for a system in the plane. In the case of the Andronov–Witt theorem, this property is called the orbital asymptotic stability of a periodic motion or attraction [15] with a specified attraction domain of trajectories: in the small (locally) or in the large (globally).

A cycle is an isolated periodic solution of an autonomous system [12]. An attracting cycle is an orbitally asymptotically stable single-frequency oscillation. In a controlled autonomous system, an oscillation is stabilized by constructing an attracting cycle.

According to [13], in the neighborhood of a cycle, the van der Pol dissipation is universal in the sense of independence from the considered system with oscillations. For a mathematical pendulum, the result established in [13] implies that any oscillation is stabilized (in the small) by using an adaptive stabilization scheme [2]. The scheme involves a control law with a parameter chosen depending on the parameter value for the oscillation to be stabilized: the control law has adaptivity. The scheme can be applied independently or as part of a more general adaptive control system.

A cycle and an attracting cycle are achieved using a feedback loop, in which a coordinate-tracking van der Pol-type controller receives trajectory information to form dissipation at the current trajectory point without delay. Thus, the stabilization problem is solved in the neighborhood of the oscillation under consideration. For global stabilization, the feedback loop is based on potential energy tracking and is described in this paper.

Other studies on the stabilization of a desired oscillation mode differ in the use of explicit time-varying control laws. Let us mention some of these studies. A review on the example of an inverted pendulum was provided in [16]. Swinging control was proposed in [17, 18]. The orbital stabilization problem of periodic solutions of low-drive nonlinear systems was solved in [19]; the nonlinear feedback control law designed therein is time-varying. Stabilization of a desired mechanical energy by impulsive control was described in [20]; a robust stabilizing control law for oscillations was found by the implicit Lyapunov method in [21]; electrodynamic control-based stabilization was carried out in [22].

In this paper, we construct a ε-corrected conservative system possessing an attracting cycle. Its attraction domain includes the oscillation domain Ω of the conservative system and the ε-neighborhood of Ω. In addition, we solve the global stabilization problem. Note that for a reduced conservative system with one degree of freedom, the local problem was solved in [2].

2. A CONSERVATIVE SYSTEM WITH ONE DEGREE OF FREEDOM. CONTROL DESIGN

Consider a smooth conservative system with one degree of freedom admitting a family Σ of single-frequency oscillations in a parameter h, where h is the constant energy value. According to [1], such a family can always be continued to a global family, so a global family Σ will be analyzed below. It occupies an oscillation domain Ω. The period on the family can be increasing (a mathematical pendulum), constant (a harmonic oscillator), or decreasing (the equation $\ddot{x} + x^3 = 0$). Under a ε-small force (a control law with a small gain ε), we obtain an autonomous corrected (controlled) conservative system of the form

$$\ddot{x} + f(x) = \varepsilon u(x, \dot{x}). \tag{1}$$

For $\varepsilon = 0$, equation (1) admits the energy integral

$$\dot{x}^2 = 2(h - \Pi(x)), \quad \Pi = \int f(x)dx. \tag{2}$$

In the domain Ω, $0 < \Pi(x) \leqslant h$. On the family Σ, the coordinate is described by the formula $x = \varphi(h, t)$ and the period $T = T(h)$ is a function of the constant energy h.

Let us choose a smooth function u without an explicit time dependence (autonomous control). In the stabilization problem, the control law must ensure local attraction to a cycle, so the function $u(x, \dot{x})$ will be of the form $u = a(x, \dot{x})\dot{x}$ [13]. The function $a = 1 - Kx^2$ itself was found in [13]; it ensures the existence of a cycle. By choosing a constant $K = K(h^*)$ in this function, one achieves the orbital asymptotic stability of the cycle born from the oscillation of a conservative system with the energy value h^* [13]. The local result becomes global for an isochronous family of oscillations. This fact is demonstrated in the van der Pol equation. However, for a nondegenerate family of oscillations, the local result does not extend to the entire domain Ω.

When solving the local problem [13], the idea is to consider the dependence of $K(h)$ on the energy value h for a nondegenerate family of oscillations. Thereby, one constructs a cycle in the corrected system close to an oscillation of a conservative system with the desired energy $h = h^*$. For this purpose, $K = K(h^*)$ is chosen.

In this paper, we apply a control law tracking in a feedback loop the current potential energy during motion. For the controlled conservative system

$$\ddot{x} + f(x) = \varepsilon[1 - K(h^*)\Pi(x)]\dot{x}, \tag{3}$$

we study the existence of a cycle close to the oscillation of the conservative system with an energy value $h = h^* > 0$. The control law is designed below as well.

Consider the amplitude (bifurcation) equation

$$I(h) \equiv \int_0^{T^*} [1 - K(h^*)\Pi(x)]\dot{x}^2 dt = 0, \tag{4}$$

in which the potential $\Pi(x)$ and kinetic $\tilde{T} = \dot{x}^2/2$ energies are calculated on the solution $x = \varphi(h, t)$. Equality (4) expresses necessary and sufficient conditions for the existence of a T^*-periodic solution in the first ε-approximation. Equality (4) takes into account the conjugate solution $\psi = -\dot{\varphi}$ of the conservative system. As it turns out, the condition $dI(h^*)/dh \neq 0$ is sufficient for the existence of a periodic solution of the perturbed equation (3); for details, see [7–9]. For the autonomous system (3), this general result means the birth of a cycle. When the derivative is negative, the cycle is stable: the formula for the CE was given in [13].

The identity

$$\int_0^T [1 - K(h)\Pi(\varphi(h,t))]\dot{\varphi}(h,t)^2 dt \equiv 0 \tag{5}$$

expresses existence conditions for a periodic solution in the first ε-approximation for all values of h. It yields the function

$$K(h) = \frac{\int_0^T (h - \Pi(\varphi(h,t)))dt}{\int_0^T \Pi(\varphi(h,t))(h - \Pi(\varphi(h,t)))dt}. \tag{6}$$

In (6), the denominator is nonzero in the oscillation domain Ω. The function $K(h)$ is defined uniquely through the potential energy Π, which varies with time t along the trajectory. Only one trajectory and one value of $K(h)$ corresponds to each h.

On the other hand, formula (5) with the upper limit of integration $T = T^*$ and the number $K(h) = K(h^*)$ leads to the amplitude equation (4) for finding the value of h corresponding to the cycle. The explicit form of the function $K(h)$ will be presented in Section 3.

Equation (3) defines a mapping of the phase plane onto itself: $t : 0 \to T$. In this case, the curve $\Gamma(h, 0) = \{x(h, 0), \dot{x}(0)\}$ is mapped into the curve $\Gamma(h, T) = \{x(h, T), \dot{x}(T)\}$. The existence of a unique root $h = h^*$ of equation (4) means the coincidence of the points $\Gamma(h^*, 0)$ and $\Gamma(h^*, T)$. This fact is observed for the curves constructed using even the first ε-approximation: the necessary and sufficient conditions in the first approximation become sufficient for the existence of a fixed point of the mapping in the first ε-approximation. For the unique root, the fixed point of the mapping will be isolated, and it corresponds to an isolated periodic solution of period T. In view of $T(h^*) = T^*$, we obtain a cycle of period T^*. The condition $dI(h^*)/dh < 0$ is valid for a contracting mapping: the cycle becomes orbitally asymptotically stable.

3. ATTRACTION TO THE CYCLE IN THE SMALL

The integral (4) is taken on the interval $t \in [0, T^*]$. Here, the function $\Pi(x)$ depends only on x and is calculated via the solution $x = \varphi(h, t)$. The change of variable $\tau = (T(h)/T^*)t$ in the integral (4) makes the coordinate x directly dependent only on τ; the function $x(h, \tau)$ becomes T^*-periodic for all values of the parameter h. For an oscillation with zero initial velocity, we obtain

$$x(h,\tau) = x(h,0)e(\tau), \quad e(\tau) = e(\tau + T^*). \tag{7}$$

The function $e(\tau)$ varies on the interval $[0, 1]$. The same is true for the function $\Pi(x(h, \tau))$. For $\tau = 0$, we have $\Pi(x(h, 0)) = h$. Therefore, due to the expression (7), the equality $\Pi(x(h, \tau)) = hz(\tau)$ holds, where a particular function $z(\tau)$ is calculated for a given potential energy. See the Appendix for a more general analysis of the application of the variable τ; Theorem 4 provided therein essentially supplements the main results, being of independent interest.

Considering the new expression for the potential energy Π, we transform the amplitude equation (4) as follows:

$$I(h) = 0, \quad I(h) \equiv h \int_0^{T^*} (1 - K(h^*)hz)(1 - z)d\tau. \tag{8}$$

The function $I(h)$ can be represented as

$$I(h) = h(\alpha - \beta h), \quad \alpha = \int_0^{T^*} (1 - z)d\tau, \quad \beta = -K(h^*) \int_0^{T^*} z(1 - z)d\tau. \tag{9}$$

Clearly, equation (8) admits the unique nonzero root $h^* = \alpha/\beta$.

Identity (5) takes the form

$$h\int\limits_0^{T^*}(1-K(h)hz)(1-z)d\tau\equiv 0.$$

Therefore, at the point $h=h^*$,

$$\frac{d}{dh}\left(h\int\limits_0^{T^*}(1-K(h^*)hz)(1-z)d\tau\right)_{h=h^*}-\frac{dK(h^*)}{dh}(h^*)^2\int\limits_0^{T^*}z(1-z)d\tau\equiv 0.$$

The derivative at the point corresponding to the cycle is calculated by the formula

$$\frac{dI(h^*)}{dh}=\frac{dK(h^*)}{dh}(h^*)^2\int\limits_0^{T^*}z(1-z)d\tau. \tag{10}$$

The function (6) is given by

$$K(h)=\frac{\int\limits_0^{T^*}(1-z(\tau))d\tau}{h\int\limits_0^{T^*}z(\tau)(1-z(\tau))d\tau}=\frac{b}{h},\quad b=\mathrm{const}>0. \tag{11}$$

The integrals in (11) are independent of the energy constant h and take positive values as sums of the integrals on quarters of the period. Therefore, $K(h)=b/h$, where $b>0$ is a constant.

Formula (11) is valid for any family of oscillations: that with an increasing (decreasing) period on the family or an isochronous family. The dependence (11) is an important characteristic of a family of oscillations.

We have the following result regarding the local stabilizability of a cycle.

Theorem 1. *The ε-corrected conservative system* (3) *always has an orbitally asymptotically stable (in the small) cycle close to the oscillation of the conservative system with the energy value $h=h^*$. This cycle attracts trajectories from its $O(\varepsilon)$-neighborhood.*

Proof. According to the amplitude equation (8), the corrected conservative system (3) has a cycle close to the oscillation of the conservative system with the energy value $h^*=\alpha/\beta$. At this point, the sign of the derivative (10) coincides with that of the number

$$\frac{dK(h^*)}{dh}=-\frac{b}{(h^*)^2}<0.$$

Therefore, by formula (10), the derivative is $dI(h^*)/dh<0$, and the mapping $t:0\to T^*$ is contracting: all trajectories from the neighborhood of the cycle are attracted to the cycle.

Remark 1. The derivative $dK(h^*)/dh$ is commonly used for proving local results on a cycle. However, the inequality $dI(h^*)/dh=-h^*\beta<0$ can be derived directly from the expression (9).

4. ATTRACTION TO THE CYCLE IN THE LARGE

For system (1) we define the total mechanical energy

$$E\equiv\frac{\dot{x}^2}{2}+\Pi(x). \tag{12}$$

This function takes a constant value h on the solutions of the conservative system. For the corrected conservative system,

$$\frac{dE}{dt} = \varepsilon[1 - K(h^*)\Pi(x)]\dot{x}^2, \tag{13}$$

where

$$\dot{x}^2 = 2(E - \Pi(x)).$$

For $\varepsilon = 0$ we have $E = h(\text{const}) > 0$.

The increment ΔE of the energy (12) on the period $T(h)$ is calculated for an oscillation with the energy value h. Therefore, for $\varepsilon > 0$,

$$\Pi(x) = \Pi(\varphi) + h\varepsilon\rho(\varepsilon, x)), \quad E - \Pi(x) = h - \Pi(\varphi) + h\varepsilon\sigma(\varepsilon, x), \tag{14}$$

where functions ρ and σ are of order one. Integration is performed over the variable τ on the interval $[0, T^*]$. As a result,

$$\Delta E(\varepsilon, h) = \varepsilon\frac{2T(h)}{T^*}[I(h) + \varepsilon hF(\varepsilon, h)], \tag{15}$$

where $\varepsilon hF(\varepsilon, h)$ is calculated by substituting the expressions (14) into equality (13) and matches the terms $h\varepsilon\rho$ and $h\varepsilon\sigma$. This formula remains valid in the entire oscillation domain Ω of the conservative system.

The function $E(\varepsilon, h^*, \tau)$ calculated on the cycle is T^*-periodic, so $\Delta E(\varepsilon, h^*) = 0$. For the cycle, we have the amplitude equation $I(h^*) = 0$. As a consequence, the equality $F(\varepsilon, h^*) = 0$ is true for the cycle.

Lemma 1. *There exists a h^*-independent number $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, the equation*

$$I(h) + \varepsilon hF(\varepsilon, h) = 0 \tag{16}$$

has a unique root.

Proof. With the representation (9), equation (16) is simplified:

$$V \equiv G(h) + \varepsilon F(\varepsilon, h) = 0, \quad G \equiv \alpha - \beta h.$$

The equation $G(h) = 0$ has the root $h^* = \alpha/\beta$ corresponding to a cycle. According to Theorem 1, the cycle is locally attracting. Hence, for $h \neq h^*$, the function $G(h)$ takes values of the same sign under small ε. The function $F(\varepsilon, h)$ vanishes at the point $h = h^*$. Therefore, the function V can be transformed to

$$V = (\alpha - \beta h)(1 + \varepsilon W(\varepsilon, h)). \tag{17}$$

Under small ε, the sign of V coincides with that of G. However, as ε increases, the second factor in (17) may vanish. The corresponding value of $\varepsilon(h)$ depends on h. For the cycle with $h = h^*$, we choose the smallest value of the number $\varepsilon(h(h^*))$ and denote by ε_0 the lower bound of the set $\{\varepsilon(h^*)\}$. Then for $0 < \varepsilon < \varepsilon_0$, equation (16) has a unique root independent of the particular value of h^*. In this case, according to (9) and (17), the existence of a root in equation (16) is determined by the term $G(h)$, which has a unique root.

The proof of Lemma 1 is complete.

Next, the corrected system (3) is investigated under $0 < \varepsilon < \varepsilon_0$. The oscillation domain Ω of the conservative system is considered; in the conservative system under analysis, there may be more than one oscillation domain.

Theorem 2. *There exists a number $\varepsilon_0 > 0$ such that the corrected conservative system* (3) *with $0 < \varepsilon < \varepsilon_0$ always has a unique cycle attracting all trajectories of the oscillation domain Ω of the conservative system.*

Proof. In the oscillation domain Ω, the corrected conservative system (3) admits (Lemma 1) a unique cycle: the corresponding value of the conservative system energy is $h = h^*$. On the cycle, $\Delta E(\varepsilon, h^*) = 0$. Beyond the cycle, the sign of the function $\Delta E(\varepsilon, h)$ coincides with that of the linear function $G(h)$, $G(h^*) = 0$. The rate of change of the function is $dG(h)/dh = -\beta < 0$. Hence, the function $G(h) \to 0$, and the trajectories of the corrected system (3) tend to the cycle. This happens with any trajectory from the oscillation domain Ω of the conservative system.

The proof of Theorem 2 is complete.

5. THE MULTIDIMENSIONAL SYSTEM

By the global family theorem [1, Theorem 1], the variables are separated in a multidimensional system. The variable x describes an oscillation family Σ on the manifold Ω invariant with respect to the phase flow of the conservative system. Outside the manifold Ω, the dynamics of the conservative system are given by the vector y of dimension $n-1$. In the domain Ω, $y \equiv 0$. Therefore, outside Ω, the motion of the conservative system in the neighborhood of the trivial solution $y = 0$ is studied.

According to Poincaré, the characteristic equation of a conservative system is reciprocal: the roots of the equation are divided into pairs containing numbers with opposite signs. Therefore, in the real-root case, under the action of ε-small control, the outgoing solutions will remain outgoing. Hence, the absence of roots with real parts is a necessary condition for attracting solutions to Ω.

Together with the separation of the integral manifold Ω, the ε-corrected system (3) with the variable x is constructed. The principal coordinates are applied for the variable y in the neighborhood of the point $y = 0$. Next, we consider a controlled conservative system of the form

$$\begin{gathered}\ddot{y}_i + k_i y_i + Y_i(y) = \varepsilon[(1 - K(h^*)\Pi(x)]\dot{y}_i,\\ k_i = \text{const}, k_i \geqslant 0, \quad i = 1, \ldots, n-1.\end{gathered} \tag{18}$$

According to (18), the energy E_y of the conservative system in the variable y varies by the law

$$\frac{dE_y}{dt} = \varepsilon[1 - K(h^*)\Pi(x)]\dot{y}^2, \quad \dot{y}^2 = \sum_{i=1}^{n-1} \dot{y}_i^2, \tag{19}$$

similar to the law (13) for the variable x. In addition, $E_y = h_y = \text{const}$ for $\varepsilon = 0$.

The energy variation law of the entire conservative system

$$\frac{d(E_x + E_y)}{dt} = \varepsilon[1 - K(h^*)\Pi(x)](\dot{x}^2 + \dot{y}^2),$$

written in the variables x and y, is associated with the amplitude equation

$$\tilde{I}(\tilde{h}) \equiv -\int_0^{T^*} [1 - K(h^*)\Pi(x)](\dot{x}^2 + \dot{y}^2)dt = 0, \quad \tilde{h} = h + h_y. \tag{20}$$

The function $\tilde{I}(\tilde{h})$ admits a simple root $\tilde{h} = h^*$, $h_y^* = 0$, which follows from the existence of a simple root in equation (4). The simple root of equation (20) is associated with a cycle of the controlled system (21) (see [8, Chap. VI, §8, p. 413, §9, p. 417]). The function $\tilde{I}(\tilde{h})$ is continuous, and the cycle of the entire system coincides with the cycle on Ω.

The laws (13) and (19) imply the equality

$$\dot{y}^2 dE = \dot{x}^2 dE_y;$$

hence, the energy in the system with the variable y changes on the period (increases and decreases) in the same way as in that with the variable x. In addition, by Theorem 2, the one-period increment $\Delta E(\varepsilon, h)$ in the variable x tends to zero whereas the trajectory on the manifold Ω to a unique cycle. Then the one-period increment $\Delta E_y(\varepsilon, h)$ in the variable y tends to zero whereas the trajectory to the unique equilibrium $y = 0$ corresponding to the cycle on Ω.

Thus, all trajectories of the domain Ω and its ε-neighborhood are attracted to the cycle. This result is true regardless of the coordinates used to define the conservative system.

Next, consider the controlled system

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = \varepsilon[1 - K(h^*)\Pi(x)]\frac{\partial \tilde{T}}{\partial \dot{q}_s}, \quad s = 1, \ldots, n, \tag{21}$$

defined by the Lagrange equations of the second kind. Here, $\tilde{T}$ and Π are the kinetic and potential energies, respectively. By assumption, for $\varepsilon = 0$ system (21) admits a family Σ of single-frequency oscillations occupying the two-dimensional domain Ω. On Ω, the system is described by the variable x; control is intended to track the potential energy on the solution. Let us choose $0 < \varepsilon < \varepsilon_0$, where ε_0 is a finite number for the oscillation family Ω; h^* is the energy value corresponding to a cycle in Ω; on Ω, the energy value $h = 0$ correspond to the equilibrium.

In a system described by Lagrange equations of the second kind, the global family of periodic motions is constructed by a continuation of its local Lyapunov family [1]. In turn, the latter is born from an equilibrium. According to the Lyapunov center theorem [23], a family of nondegenerate local nonlinear periodic solutions exists in system (18) if, in addition to pure imaginary roots, this system has non-resonant frequencies $\sqrt{k_j} \neq p\sqrt{k_s}$, $p \in \mathrm{N}$. Moreover, by the Lyapunov center theorem [23], the Lyapunov family always exists for the largest frequency. In the analysis of system (18), the above conditions have not been imposed. They arise in the Lagrangian system. Keeping this aspect in mind, we proceed to the controlled system (21).

When the principal coordinates for the system in y are not separated, the right-hand sides of system (18) will contain linear combinations of the velocities $\dot{y}_i$; they are partial derivatives of the kinetic energy $\tilde{T}$ of the system in $\dot{y}_s$. The reconstruction of system (21) is completed by returning to the initial stage of building the reduced system with one degree of freedom [1]. At this stage, the velocity $\dot{y}_i$ with a constant factor is added to the linear combination of the velocities $\dot{y}_i$.

When formulating Theorem 3, we accept the hypotheses of the Lyapunov theorem about the center adjacent to the oscillation domain Ω.

Theorem 3 (on the corrected conservative system). *Let a conservative system described by the Lagrange equation of the second kind admit an oscillation family with a domain* Ω. *Then there exists a number* $\varepsilon_0 > 0$ *such that the corrected system* (21) *with* $0 < \varepsilon < \varepsilon_0$ *always has a unique cycle in* Ω *that is* ε*-close to the oscillation of the conservative system with energy* $h = h^*$. *The cycle attracts trajectories from the oscillation domain* Ω *and also those* ε*-close to* Ω. *On the solutions of the corrected system, the energy variation law* $E = \tilde{T} + \Pi$ *of the conservative system is given by*

$$\frac{dE}{dt} = \varepsilon[1 - K(h^*)\Pi(x)]\tilde{T}.$$

Remark 2. The theorem on a corrected conservative system is formulated for a Lagrangian system. The application of Theorem 3 to a conservative system written in other variables is demonstrated in the examples below.

6. SOME EXAMPLES

First, we analyze the rate of approaching the cycle in energy terms. On the trajectories of the corrected conservative system (3), the energy variation law is given by (13). The one-period energy increment is calculated by formula (15). For a given $\varepsilon < \varepsilon_0$, it equals $2T(h)hV$. On the other hand, for an energy value h, the cycle is approached with a rate V almost linear in h. For $h > h^*$, the attraction to the cycle occurs with increasing rate V, which is negative on the above interval. For trajectories with an initial positive energy, the rate V is positive all the time while approaching the cycle. At the point $h^* = \alpha/\beta$, the rate V changes its sign from plus to minus. Let us provide several examples.

Example 1. The van der Pol equation in the adaptive stabilization scheme is described by

$$\ddot{x} + x = \varepsilon(1 - K(h^*)x^2)\dot{x}, \quad K(h^*) = \mathrm{const} > 0. \tag{22}$$

In the van der Pol equation, $K(h^*) = 1$.

For $\varepsilon = 0$, we have a harmonic oscillator with the potential energy $\Pi = x^2/2$. The generating oscillation is given by $x = A\cos t$, where A denotes an amplitude. The energy is $h = A^2/2$; therefore, $\tau = 2t$, $\Pi = hz(\tau)$, and $z = 1 + \cos 2t$. Calculations by formula (11) yield the number $b = 2$. The amplitude for the cycle is $A^* = 2/\sqrt{K(h^*)}$.

Equation (22) satisfies all the hypotheses of Theorem 2. Therefore, the corrected linear oscillator has a globally attracting cycle that passes, depending on $K(h^*)$, through any given point of the phase plane. The oscillation domain Ω in equation (22) coincides with the entire phase plane, excluding the punctured zero. Far from the cycle, as well as near zero, the rate of approaching the cycle is proportional to εh^2. Near the cycle with energy h^*, this rate equals $\varepsilon h|\Delta h|$, $|\Delta h| = h - h^*$.

In equation (22) with $K = 1$, the cycle was constructed by van der Pol [4] and, independently of him, by Andronov [11].

Note that the doubled potential energy is applied in the van der Pol oscillator. In the local ε-theory, the number ε^* is not estimated; the cycle remains attracting when increasing ε; the farther ε is from zero, the lesser the oscillator's behavior will resemble harmonic oscillations.

Example 2. Consider the corrected mathematical pendulum

$$\ddot{x} + \sin x = \varepsilon(1 - 2K(h^*)\sin^2(x/2))\dot{x}.$$

In a nonlinear system, the calculation of the function $z(\tau)$ is complicated due to the unknown function describing the oscillations. For a mathematical pendulum, $\Pi = 2\sin^2(x/2)$. The dependence (11) was numerically obtained in [24].

The peculiarity of the mathematical pendulum is that the oscillation domain has boundedness from above in h. Near the point $h = 0$ (small oscillations), the rate of approaching the cycle is proportional to εh^2.

The mathematical pendulum is investigated in the relative motion problem of a satellite in the plane of a circular orbit [25]. An adaptive scheme for stabilizing the satellite oscillation (in the small), given by

$$\ddot{x} + |\mu|\sin x = \varepsilon\sigma(1 - 2K(h^*)x^2)\dot{x}, \tag{23}$$

was proposed in [2]; also, the local attraction problem was solved for any trajectories from the neighborhood of the stabilized oscillation.

Let us introduce the following notation for the satellite [25]: μ is the inertial parameter ($|\mu| \leqslant 3$); α is the angle between the radius vector of the center of mass and the main central axis of inertia of the satellite in the orbital plane; v is the true anomaly chosen as the independent variable. In equation (23), $x = 2\alpha$ and $\mu > 0$ or $x = 2\alpha + \pi$ and $\mu < 0$, and $\sigma = 1$.

By replacing the term $2K(h^*)x^2$ in equation (23) with $2K(h^*)|\mu|\sin^2(x/2)$, we achieve a cycle attracting all trajectories from the oscillation domain (see Theorem 2). Thus, the global attraction of trajectories is ensured.

For small μ (a "flattened" satellite), it may be interesting to use the stabilized long-periodic oscillation of the satellite (in the large) near the equilibrium instead of the latter.

Example 3. The two-body problem

$$\frac{d^2x}{dt^2} = -\frac{\gamma x}{x^2+y^2}, \quad \frac{d^2y}{dt^2} = -\frac{\gamma y}{x^2+y^2}, \quad \gamma > 0,$$

has the area integral

$$x\frac{dy}{dt} - y\frac{dx}{dt} = c, \quad c = \text{const}. \tag{24}$$

On the integral manifold (24), the dynamics are described by the conservative system

$$\frac{d^2r}{dt^2} = \frac{c^2}{r^3} - \frac{\gamma}{r^2}, \quad r^2 = x^2 + y^2,$$

which possesses a family of elliptic orbits for a fixed value $c = c_*$. (The constant solution of this equation is associated with circular orbits.)

According to Theorem 2, in the corrected system

$$\frac{d^2r}{dt^2} - \frac{c_*^2}{r^3} + \frac{\gamma}{r^2} = \varepsilon((1 - K(h^*)\Pi(r))\frac{dr}{dt}, \quad \Pi = \frac{2c_*^2}{r^2} - \frac{\gamma}{r},$$

any orbit of the two-body problem close to the elliptic orbit with energy h^* is stabilized (in the large). The attraction of other orbits to the plane c_* is ensured by the equation $\dot{\Delta}c = -\Delta c$, $\Delta c = c - c_*$.

Example 4. The dynamics of a heavy solid with a fixed point are described by the classical Euler–Poisson equations

$$\begin{aligned} A\dot{p} &= (B-C)qr + P(z_0\gamma_2 - y_0\gamma_3), & \dot{\gamma}_1 &= \gamma_2 r - \gamma_3 q, \\ B\dot{q} &= (C-A)rp + P(x_0\gamma_3 - z_0\gamma_1), & \dot{\gamma}_2 &= \gamma_3 p - \gamma_1 r, \\ C\dot{r} &= (A-B)pq + P(y_0\gamma_1 - x_0\gamma_2), & \dot{\gamma}_3 &= \gamma_1 q - \gamma_2 p, \end{aligned} \tag{25}$$

written in quasi-coordinates: A, B, and C are the principal moments of inertia of the body; P is the body weight; x_0, y_0, and z_0 are the coordinates of the center of gravity; $\Omega = (p, q, r)$ is the angular velocity; finally, $\Gamma = (\gamma_1, \gamma_2, \gamma_3)$ is the unit vertical vector directed upward.

For $y_0 = 0$, system (25) admits the integral manifold

$$p = 0, \quad r = 0, \quad \gamma_2 = 0. \tag{26}$$

The Mlodzeevskii motions [26] realized on this manifold are described by

$$B\dot{q} = P(x_0\gamma_3 - z_0\gamma_1), \quad \dot{\gamma}_1 = \gamma_2 r - \gamma_3 q, \quad \dot{\gamma}_3 = \gamma_1 q - \gamma_2 p. \tag{27}$$

Using the geometric relation $\gamma_1^2 + \gamma_3^2 = 1$ and the changes $\gamma_1 = \cos\delta$ and $\gamma_2 = \sin\delta$, we reduce system (27) to the mathematical pendulum equation

$$B\ddot{\delta} + P\sqrt{x_0^2 + z_0^2}\sin(\delta - \nu) = 0, \quad \nu = \tan(z_0/x_0).$$

Thus, for any energy value chosen, an attracting cycle (in the large) is constructed for the Mlodzeevskii oscillation. As in Example 3, the attraction to the manifold (26) is ensured by linear dissipation in the variables p, r, and γ_2.

Together with the planar Mlodzeevskii oscillations, the body under study admits a second family of pendulum oscillations [27]. Being spatial, this family is described by a reduced conservative system with one degree of freedom and is separated, step by step, from system (25) with $y_0 = 0$ when constructing the reduced system. Stabilization of the oscillations (in the large) is performed according to the theory of Sections 2–4.

Remark 3. These examples have presented new results for the corresponding problems.

7. CONCLUSIONS

In a conservative system, oscillations form families. Therefore, an oscillation can be stabilized only within a controlled system. Control laws with an explicit time dependence are commonly used.

For a conservative system admitting an oscillation family, a ε-corrected system with an attracting cycle (in the large) is always constructed. An autonomous controller with a small gain is applied. It is given by nonlinear dissipation that acts without delay at the current trajectory point and tracks the potential energy of the system. The attraction domain of the cycle includes the oscillation domain Ω of the conservative system and the ε-neighborhood of Ω. The cycle is constructed for an oscillation with any desired energy of the conservative system. Stabilization is performed according to the adaptive scheme.

The main results of this study have been formulated in three theorems. Theorem 1 shows the existence of a cycle and provides a solution of the stabilization problem in the small (in the neighborhood of the oscillation considered). Next, Theorem 2 establishes the attraction of trajectories evolving from any point of the oscillation domain of the conservative system to the cycle. Theorem 3 extends the results of Theorems 1 and 2 (for a system with one degree of freedom) to the multidimensional case, including the corresponding corrected Lagrangian system. New results in classical problems have been annotated in the examples.

The paper has settled several issues in nonlinear mechanics, oscillation theory, bifurcation, and control theory. In classical mechanics, only linear dissipation is considered. This study gives an example of universal nonlinear dissipation defined by potential energy. It can also explain phenomena in nature.

To investigate the family of nonlinear oscillations of a conservative system, we have proposed to apply the theory of linear systems. The corresponding result (Theorem 4) is postponed to the Appendix, finalizing the author's efforts to present the main material in a comprehensible way. The idea of introducing *new time* brings the system of nonlinear oscillations to an isochronous family.

The conclusions on attraction in the large have become qualitatively new in bifurcation theory: a small parametric perturbation of a system leads to a global rearrangement of its phase portrait.

Concerning control theory, we have suggested the idea of utilizing the nature of an unclosed system. The control law designed is only corrective. In Examples 3 and 4, the stabilization scheme has been demonstrated on classical problems. Other applications include the problems of orbital stabilization of spacecraft in long-range missions. Here, the main constraint is the energy resource. Therefore, maneuvers using gravitational attraction (potential energy) are in demand.

APPENDIX

Transforming a family of nondegenerate (nonlinear) oscillations to an isochronous family

Consider the conservative system

$$\ddot{x} + f(x) = 0, \quad \frac{\dot{x}^2}{2} + \int f(x)dt = h(\text{const}),$$

admitting a family Σ of nondegenerate single-frequency oscillations. On the nondegenerate family, by definition, the period $T(h)$ varies monotonically with h.

Theorem 4. *The nondegenerate oscillation family $\Sigma = \{x(h,t)\}$ of a conservative system is always transformed to an isochronous family with time $\tau = (T(h)/T^*)t$, which is chosen together with the period T^* of the family oscillation, the potential energy $\tilde{\Pi}(x) = \Pi(x)(T^*/T(h))^2$ and the total energy $\tilde{h} = h(T^*/T(h))^2$. On the family Σ, the law*

$$x^2\left(h\left(\frac{T^*}{T(h)}\right)^2, 0\right) = h\left(\frac{T^*}{T(h)}\right)^2$$

relates the amplitude and energy of the oscillations.

Proof. The potential energy $\Pi(x)$ is a function of one variable x. It depends on h through the solution $x = \varphi(h,t)$. When passing to the new independent variable $\tau = (T(h)/T^*)t$, the period on the oscillation becomes T^*, so x is a T^*-periodic function of the variable τ. This occurs for all oscillations of the family, which is thus transformed to an isochronous family with period T^*. For an oscillation with zero initial velocity, we have $\Pi(x(h,0)) = h$. The initial value h is preserved in the function $x(h,\tau)$; therefore,

$$\Pi = hz(\tau), \quad 0 \leqslant z(\tau) \leqslant 1, \quad z(\tau) = z(\tau + T^*).$$

With the new independent variable τ, the energy integral on the family of oscillations takes the form

$$\left(\frac{T(h)}{T^*}\right)^2 \left(\frac{dx}{d\tau}\right)^2 = 2(h - \Pi). \tag{A.1}$$

In a more conventional representation, we obtain

$$\left(\frac{dx}{d\tau}\right)^2 = 2(\tilde{h} - \tilde{\Pi}), \quad \tilde{h} = h\left(\frac{T^*}{T(h)}\right)^2, \quad \tilde{\Pi} = \Pi(x)\left(\frac{T^*}{T(h)}\right)^2. \tag{A.2}$$

The oscillations with initial zero velocity in the transformed system are described by

$$x(\tau) = x(\tilde{h}, 0)e(\tau), \quad 0 \leqslant e(\tau) \leqslant 1, \quad e(\tau) = e(\tau + T^*).$$

On these oscillations, the law

$$x^2(\tilde{h}, 0) = \tilde{h} \tag{A.3}$$

expresses the dependence of the amplitude of isochronous oscillations on the system energy $\tilde{h}$. Hence, the amplitude of nonlinear oscillations on the family Σ depends on the energy h according to the law

$$x^2\left(h\left(\frac{T^*}{T(h)}\right)^2, 0\right) = h\left(\frac{T^*}{T(h)}\right)^2. \tag{A.4}$$

Remark 4. In the case of a linear oscillator in (A.3), the function $T(h)$ reduces to a constant and the dependence (A.3) is known.

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